HARVARD SUMMER SCHOOL

The Harvard Summer School Economics Program offers a variety of courses, ranging from introductory economics to advanced classes in micro- and macroeconomic theory, econometrics, international economics, and finance.

Advanced courses presuppose analytical and technical facility. These courses are designated as starred courses and appear in the Summer School catalogue with asterisks before their numbers (e.g. *ECON S-1010). **All students enrolling in starred economics courses must pass the proficiency examination offered at the beginning of the Summer School**, unless the Economics Program office records show that they have passed it in a previous year. Students who do not meet this requirement will be automatically withdrawn from starred courses. (Harvard College students should see the relevant section in the Summer School catalogue for term-time courses that exempt them from the proficiency examination.)

The proficiency examination will be held on **Monday and Tuesday of the first week** of classes and will last approximately 90 minutes. The place and time are in the Summer School catalogue. We encourage you to refresh your mathematical skills and your understanding of the fundamental tools of economic analysis in preparation for the exam. Students should be familiar with college algebra, graphing functions, differentiation (including partial derivatives and applications in optimization), integration, exponential and logarithmic calculus, and basic statistics. Moreover, students should be prepared to apply these concepts to problems in economics.


The enclosed sample questions will also give you an idea of what to expect on the exam. Results will be available on Wednesday at 10 a.m., which will give students who fail the test time to switch from starred courses to less technically demanding courses.

**NOTE:** For purposes of maximum legibility and clarity, it is recommended that you print the sample exam and suggested solution.
Sample Questions for PROFICENCY EXAM
Harvard Summer School -- Economics Program

Students may not use calculators at the proficiency examination in June.

1. Mr. B has $20.00 to spend on fruit. Apples cost $0.25 apiece. Oranges cost $0.50 apiece.

   a. Write down a formula for the maximum number of apples Mr. B can buy as a function of the number of oranges he purchases.

   b. In a graph with the quantity of oranges measured on the horizontal axis and the quantity of apples measured on the vertical axis, indicate the area that represents all the combinations of apples and oranges that Mr. B can afford to purchase.

2. Graph the equation \( y = 30 - x^2 + 2x \).

3. Ms. Marksman is standing in the x-y plane at the coordinates (50,70). She shoots a bullet through a paper target located at (200,10). The bullet travels in a straight line, hits the target, and continues traveling at a constant speed.

   a. Write down the formula for the straight line within which the bullet travels.

   b. If the bullet hits the target six seconds after Ms. Marksman fires her gun, what will be the bullet's coordinates one second after passing through the target?

4. Mr. Farmer has a triangular garden with comers at (0,0), (50,20) and indicates a distance in meters from the (200,0). Each coordinate indicates a distance in meters from the origin. If it costs $2 to irrigate one square meter of a garden for one week, what will be Mr. Farmer's total irrigation bill for the ten week dry season?

5. Suppose that demand in the market for shirts is given by the formula \( p = 100 - 2Q_d \) where \( p \) is the price per shirt, \( Y \) is aggregate weekly consumer income, and \( Q_d \) is the quantity of shirts demanded per week.
Suppose that the supply for shirts is given by

\[ p = 20 + 6Q_s \]

where \( Q_s \) is the quantity of shirts supplied per week.

a. At what price is the quantity supplied equal to the quantity demanded?

b. When quantity supplied equals quantity demanded, what is the total weekly spending on shirts in this market?

c. Graph the supply curve and the demand curve together in a single diagram with price on the vertical axis and quantity on the horizontal axis. Indicate the price at which \( Q_s = Q_d \). Indicate the total spending on shirts.

d. Suppose the price of shirts is set by law at \( p = 50 \). Calculate the difference between the quantity demand and quantity supplied at that price.

6. What is the slope of the line tangent to the curve \( y = x^2 \) at the point \((4, 16)\)?
8. Calculate \( \frac{d^2y}{dx^2} \) when
   
   a. \( y = 4x^4 + 3x^2 + 8 \)
   b. \( y = 8 \ln x \)
   c. \( y = x^{0.25} \)

9. Let \( y = \sqrt{x_1 \cdot \ln x_2} \). Assume \( x_1 \) and \( x_2 \) take only positive values. Calculate each of the following partial derivatives:
   
   a. \( \frac{\partial y}{\partial x_1} \)  
   b. \( \frac{\partial^2 y}{\partial x_1^2} \)  
   c. \( \frac{\partial y}{\partial x_2} \)  
   d. \( \frac{\partial^2 y}{\partial x_2^2} \)  
   e. \( \frac{\partial^2 y}{\partial x_2 \partial x_1} \)  
   f. \( \frac{\partial^2 y}{\partial x_1 \partial x_2} \)

10. Suppose that consumer spending plans are described by the function
    
    \[ C = c_0 + c_1 \cdot Y \]
    
    where \( C \) is total consumer spending, \( Y \) is income and \( c_0 \) and \( c_1 \) are positive constants, and \( c_1 < 1 \). Suppose also that business investment spending plans are given by the function
    
    \[ I = i_0 - i_1 \cdot r \]
    
    where \( I \) is total business investment spending, \( r \) is the interest rate and \( i_0 \) and \( i_1 \) are positive constants. Finally, suppose that the economy is in equilibrium when total planned spending equals total income, that is when
    
    \[ Y = C + I \]
    
    a. Find an expression for the economy's equilibrium income level.
    
    b. Find an expression that describes how equilibrium income changes as \( i_0 \) varies.
    
    c. Graph equilibrium income as a function of the interest rate in a graph with \( r \) on the vertical axis and \( Y \) on the horizontal. What is the slope of this function?
11. Suppose that a firm's production function takes the form

\[ y = K^{0.5} \cdot L^{0.5} \]

where \( y \) indicates total output per week, \( K \) is capital input (machine hours per week) and \( L \) is labor input (worker hours per week).

Suppose that \( K = 625 \).

a. Graph output as a function of labor input with output on the vertical axis and labor on the horizontal axis.

b. Find the rate at which output increases as additional labor is added. The formula you derive is called the marginal product of labor and will be a function of total labor input. Draw a graph of this function.

c. Suppose that labor costs $10 per unit (per worker hour). What is the total labor cost if the firm is producing 100 units of output?

d. Suppose that capital costs $4 per unit. Write a formula that expresses the total weekly cost of production as a function of total output produced.

e. Find the formula that gives the rate of increase of total production costs as output increases. This formula — called the firm's marginal cost of production — will be a function of the level of output.

12. Suppose that a firm's production function takes the form

\[ y = K^{0.5} \cdot L^{0.5} \]

a. In a graph with \( K \) on the vertical axis and \( L \) on the horizontal axis, draw a curve showing all combinations of \( K \) and \( L \) that can produce 120 units of output.

b. Write the formula for this curve with \( K \) as a function of \( L \).

c. Find the slope of the line tangent to this curve at (36,400).
d. Use a derivative to answer this question: Approximately how much extra capital must you employ to replace one unit of labor if you start with $K = 100, L = 144$?

13. Suppose that $y = F(K, L)$.
Suppose also that $\frac{\partial F}{\partial K}$ and $\frac{\partial F}{\partial L}$ are well defined and non-zero for all values of $K, L$. Pick a constant $y_0$, and draw the curve $y_0 = F(K, L)$, as in the graph at right. What is the slope of the line tangent to the curve at the point $(K_0, L_0)$?

14. Suppose that the production technology for corn is given by

$$C = 2\sqrt{L_c}$$

where $C$ is the output of corn (in bushels) and $L_c$ is the quantity of land (in hectares) devoted to corn production. Suppose also that the production technology for rice is given by

$$R = \ln L_r$$

where $R$ is the output of rice and $L_r$ is the quantity of land devoted to rice production. Suppose a particular economy has 1000 hectares of land to devote to the production of grain.

a. If the economy produces (ln 900) bushels of rice, how much corn can it produce?

b. Write a formula expressing the maximum amount of rice the economy can produce as a function of its corn output.

c. Use a derivative to answer this question: Approximately how much rice will this economy have to give up to increase corn output by 1 bushel if its rice output is initially Ln 900?
15. Suppose that the firm EcoStar owns a section of jungle land where oil has been discovered. It costs nothing to pump the oil from the ground, but the drilling pollutes the jungle. The total amount of pollution created \((D)\) is given by the equation \(D = \frac{1}{2} B^2\) where \(B\) is the total number of barrels of oil produced. EcoStar can sell each barrel of oil for a price of \$1000.

Suppose also that 100 tourists visit the jungle each year. EcoVisa can charge a recreation fee, and each tourist will pay 1000 - \(D\) for the right to explore the jungle, where \(D\) is the total amount of pollution from oil drilling.

a. How many barrels of oil should EcoVisa produce if it wants to maximize its combined profits from oil sales and tourist revenues?

b. Write down the condition that assures you that your answer in part (a) is a maximum and not a minimum.

16. Mr. Farmer has 20 feet of wire and wants to enclose a rectangular garden plot with the maximum possible area. How large an area can he enclose?

17. You are looking at the aerial view of a hillside. The vertical coordinates \((y)\) indicate latitude, the horizontal coordinates \((x)\) indicate longitude. The altitudes in meters at any point is given by the formula

\[
A = \sqrt{x \cdot y}
\]

where \(A\) is the altitude of the point at latitude \(x\) and longitude \(y\). For instance, at \((27,3)\) the altitude is \(\sqrt{81} = 9\).

a. Graph the curve connecting all points \((x,y)\) where the altitude is exactly 60 meters.

b. Write a formula for the slope of a line tangent to any given point \((x,y)\) on this curve.

c. Suppose a fence runs across this hillside from the point \((0,100)\) to \((200,0)\). From the air (in the \(x\)-\(y\) plane), this fence looks like a straight line. At what coordinates does this fence reach its highest altitude?
18. Suppose your business expects a continuous profit flow beginning now and lasting for 20 years at the constant rate of $10,000 per year. Suppose you discount this stream of earnings at a (continuously compounded) annual rate of 5%. Write the formula for the present value of this 20-year stream of earnings.

19. You have the following paired observations of two variables, x and y:

\[
\begin{align*}
  x_1 &= 10, & y_1 &= 6 \\
  x_2 &= 4, & y_2 &= 8 \\
  x_3 &= 12, & y_3 &= 2 \\
  x_4 &= 16, & y_4 &= 4 \\
  x_5 &= 8, & y_5 &= 10 \\
\end{align*}
\]

a. Calculate the mean of the x observations.
b. Calculate the variance of the y observations.
c. Calculate the standard deviation of the x observations.
d. Calculate the covariance of (X,Y). Assume that each of the (x,y) pairs listed occurs with equal probability.

20. Suppose you toss a fair coin n times.

a. What is the probability of tossing all heads and no tails?
b. What is the probability of tossing at least one heads?

21. Suppose that the mean height in a population of students is normally distributed with a mean of 1.8 meters and a standard deviation of ten centimeters (0.1 meters). Use the attached normal distribution to calculate the following probabilities:

a. Probability that a student selected at random will be more than 2 meters tall.
b. Probability that a randomly-selected student will be between 1.7 and 2 meters tall.
c. The probability that if you select 10 students at random, at least one of them will be more than 2 meters tall.
### Table For $N(x)$ When $x \leq 0$

This table shows values of $N(x)$ for $x \leq 0$. The table should be used with interpolation. For example

\[ N(-0.1234) = N(-0.12) - 0.34(N(-0.12) - N(-0.131)) \]
\[ = 0.4522 - 0.34 \times (0.4522 - 0.4483) \]
\[ = 0.4509 \]

<table>
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<th>$x$</th>
<th>00</th>
<th>01</th>
<th>02</th>
<th>03</th>
<th>04</th>
<th>05</th>
<th>06</th>
<th>07</th>
<th>08</th>
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</thead>
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<td>0.4994</td>
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<td>0.4975</td>
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<td>0.4996</td>
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<td>0.4980</td>
<td>0.4976</td>
<td>0.4973</td>
<td>0.4969</td>
</tr>
</tbody>
</table>

### Table For $N(x)$ When $x \geq 0$

This table shows values of $N(x)$ for $x \geq 0$. The table should be used with interpolation. For example

\[ N(0.6278) = N(0.52) + 0.78[N(0.53) - N(0.52)] \]
\[ = 0.7324 + 0.78 \times (0.7357 - 0.7324) \]
\[ = 0.7350 \]
Short Answers to Sample Questions
for the Proficiency Exam

Harvard Summer School Economics Program

Note: References in parentheses are to relevant pages in the standard texts:
Chiang, Alpha, *Fundamental Methods of Mathematical Economics* (3rd ed.)
Larsen, Richard and Morris Marx, *An Introduction to Mathematical Statistics and Its Applications*
(2nd ed.) [Designated as L&M]

1. a. \( \Delta = 80 - 2 \Omega \) where \( \Delta = \) apples, \( \Omega = \) oranges
b. Graph is a straight line with vertical intercept 80, horizontal intercept 40; area of all affordable bundles is entire area between this line and the two axes.
   (Chiang, p. 24)

2. Graph is a concave parabola with maximum at \( x = 1, y = 31 \) (Chiang, p. 41)

3. a. \( y = 90 - 2/5 \, x \)
   b. \( (225, 0) \)
   (Chiang, p. 74)

4. $40,000

5. a. \( P' = 80 \)
   b. Since \( Q' = 10, P'Q' = 800 \)
   c. \( Q_1 - Q_2 = 20 \)
   (Chiang, pp. 46-51)

6. \( \frac{dy}{dx} = 8 \) at \( (4, 16) \) (Chiang, p. 131)

7. a. \( \frac{df}{dx} = \frac{1}{x}; \) at \( x = \frac{1}{8}, \frac{df}{dx} = 8 \) (Chiang, p. 292)
   b. \( \frac{df}{dx} = 4 \) at all \( x \) (Chiang, p. 155)
   c. \( \frac{df}{dx} = \frac{1}{18} \, x^{1/2}; \) at \( x = \frac{1}{9}, \frac{df}{dx} = \frac{1}{6} \) (Chiang, p. 155)
   d. \( \frac{df}{dx} = 10e^x; \) at \( x = 0, \frac{df}{dx} = \frac{10}{x} \) (Chiang, p. 293)
   e. \( \frac{df}{dx} = -2x^3; \) at \( x = -4, \frac{df}{dx} = \frac{1}{8} \) (Chiang, p. 155)
   f. \( \frac{df}{dx} = \frac{1}{2} \, x^3 + \frac{1}{3} \, x^{2/3}; \) at \( x = 8, \frac{df}{dx} = \frac{32 \, 1}{12} \) (Chiang, p. 156)

8. a. \( \frac{dy}{dx^2} = 48x^2 + 6 \)
   b. \( \frac{dy}{dx^2} = -8x^2 \)
   c. \( \frac{dy}{dx^2} = .1875x^{1.5} \)
   (Chiang, p. 239)

9. a. \( \frac{\partial y}{\partial x_1} = \frac{1}{2} \, \frac{1}{2} \, (\ln x_2)^{\frac{1}{3}} \)
   d. \( \frac{\partial y}{\partial x_2} = \frac{1}{2} \, \frac{1}{2} \, (\ln x_2)^{-\frac{1}{2}} \, \frac{1}{x_2} \)
   e. \( \frac{\partial y}{\partial x_2} = \frac{1}{2} \, x_1 \, \frac{1}{2} \, (\ln x_2)^{-\frac{1}{2}} \, \frac{1}{x_2} \)
   f. \( \frac{\partial y}{\partial x_1} = \frac{1}{2} \, x_1 \, \frac{1}{2} \, (\ln x_2)^{-\frac{1}{2}} \, \frac{1}{x_2} \)

   (Chiang, p. 174)
10. a. \[ Y^* = \frac{1}{1-c_i} (c_{o+1}^* - i - i_f) \quad \text{(Chiang, p. 52)} \]

b. \[ \frac{\partial Y^*}{\partial i_0} = \frac{1}{1-c_i} \quad \text{the multiplier} \quad \text{(Chiang, p. 52)} \]

c. The slope is given by \[ \frac{dr}{dY^*} = \frac{c_{i+1} - 1}{i_1} \]

11. a. Graph is a concave function from the origin.

b. \[ MP_L = \frac{dy}{dL} - \frac{25}{2} L^{-\frac{1}{2}} \] \quad Graph is a convex function.

c. \[ \text{S160} \quad \text{d. TC} = 2500 + \frac{2}{125} y^2 \quad \text{e. MC} = \frac{d(TC)}{dy} = \frac{4}{125} y \]

(Chiang, p. 176)

12. a. Graph is a convex isoquant.

c. \[ \frac{dK}{dL} = \frac{100}{9} \]

b. \[ K = \frac{120^3}{L} \]

d. Since \[ \frac{dK}{dL} = \frac{-K}{L}, \] then for \( K = 100, \ L = 144, \)

\[ \frac{dK}{dL} = \frac{-100}{144} = -\frac{25}{36}. \] Hence, to replace 1 unit of \( L \) requires 25/36 units of \( K. \)

(Chiang, p. 360)

13. Slope \[ \frac{dK}{dL} = \frac{\partial F/\partial L}{\partial F/\partial K} \quad \text{(Chiang, p. 194)} \]

14. a. \[ C = 20 \]

b. \[ K = \ln[1000 - \left(\frac{C}{2}\right)^2] \] \quad the production possibilities frontier.

c. \[ \frac{dR}{dC} = \frac{1}{1000 - \left(\frac{C}{2}\right)^2} \left(\frac{1}{2} C\right) \quad \text{hence, at } K = \ln900, \ \text{dR/dC} = -1/90; \text{ if C increases by 1 bushel,} \]

\( R \) production must decrease by 1/90th of a bushel.
15. a. $B^* = 10$ barrels  
   b. at $B^* = 10$, \( \frac{d^2x}{dB^2} = -100 < 0 \), the second order condition.  
   (Chiang, pp. 231, 259)

16. Set up a constrained maximization with Lagrangeans—max. area = 25. (Chiang, p. 372)

17. a. Graph is a convex curve  
   c. (100, 50)
   b. \( \frac{dy}{dx} = \frac{-\partial F/\partial x}{\partial F/\partial y} = \frac{1}{2x^2} \cdot \frac{1}{2y^2} = \frac{-1}{2x^2y^2} \)  
   (Chiang, p. 372)

18. $PDV = \int \sum_{i=1}^{20} \int \sum_{j=1}^{10,000} e^{-\alpha t} dt$  
   (Chiang, p. 463)

19. a. Sample mean of $x = \bar{x} = 10$  
   (L&M, p. 166)
   b. Sample variance of $y = s_y^2 = \frac{1}{n-1} \sum_{i=1}^{n} (y_i - \bar{y})^2 = \frac{40}{4} = 10$  
   (L&M, p. 240)
   c. Sample standard deviation $s_x = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2} = \sqrt{\frac{80}{4}} = \sqrt{20} = 2\sqrt{5}$  
   (L&M, p. 240)
   d. Sample Cov(x, y) $= \frac{1}{n} \sum_{i=1}^{n} x_i y_i - \bar{x} \bar{y} = \frac{1}{5} (60 + 32 + 24 + 64 + 80) - (10)(6) = -8$  
   (L&M, pp. 436-7)

20. a. $P(\text{All Heads}) = (1/2)^n$  
   b. $P(\text{At Least 1 Head}) = 1 - (1/2)^n$  
   (L&M, Chap. 2)

21. a. 0.0228  
   b. 0.5185  
   c. 0.2060  
   (L&M, Sec. 4.3)